Self trapped gravitational waves (geons) with anti-de Sitter asymptotics

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ELTE, 20 March 2017

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1. Asymptotically flat geons
2. Anti-de Sitter spacetime (AdS)
   - instability
3. Asymptotically AdS spacetimes
   - conserved quantities
4. AdS geons
   - nonlinear perturbative construction based on spherical harmonic decomposition
   - helically symmetric rotating geons
   - comparison to numerical results
Introduction: asymptotically flat geons


*Figure 1. John Wheeler lecturing* at a conference in Cambridge, UK, in 1971. Wheeler's style was to cover the blackboard with inspirational colored-chalk diagrams and phrases before the lecture, then work his way through them, one by one.
Toroidal geons

Held together by the gravitational attraction of the mass associated with the electromagnetic field energy

Wheeler studied very thin torus and very high wave numbers
Small amplitude high frequency high angular momentum waves going around in a circle

Brill and Hartle,
*Phys. Rev.* 135, B271 (1964): similar geons formed by vacuum gravitational waves
Spherical geons: electromagnetic or gravitational waves

Brill and Hartle 1964:

Large number of identical-size thin toroidal geons with different orientations

The metric becomes spherically symmetric and static on the large scale

$$ds^2 = -e^{\nu} dt^2 + e^\lambda dr^2 + r^2 d\Omega^2$$

There is a thin sphere active region, where the high frequency waves are concentrated – inner region flat, outside Schwarzschild

For $\Lambda < 0$ the spacetime of geons must tend asymptotically to the anti-de Sitter metric (AdS)

$\Lambda < 0$ provides an effective attractive force

– formation of localized solutions is easier

Localized linear perturbative modes of AdS are studied by


There are one-parameter families of geon solutions emerging from the linear modes

Global spatially compactified coordinates

\[ ds^2 = \frac{L^2}{\cos^2 x} \left( -dt^2 + dx^2 + \sin^2 x \, d\Omega^2 \right) \]

where \( L^2 = -3/\Lambda \)

- each point corresponds to a 2-sphere with radius \( L \tan x \)
- metric is static in these coordinates
- center is at \( x = 0 \), infinity at \( x = \frac{\pi}{2} \)
- range of time coordinate: \( -\infty < t < \infty \)
- radial outwards acceleration of constant \( x \) observers is \( \frac{\sin x}{L} \)
- timelike geodesics meet again at a point
Cosmological-type coordinates
\[ ds^2 = L^2 \left[ -d\tau^2 + \cos^2 \tau \left( d\rho^2 + \sinh^2 \rho \, d\Omega^2 \right) \right] \]

- Friedmann-Lemaître-Robertson-Walker universe with \( k = -1 \)
- constant \( \tau \) slices are homogeneous hyperbolic spaces
- constant \( \rho \) lines are geodesics
- coordinate singularity at \( \tau = \pm \pi/2 \)

Review of AdS coordinate systems in:

We use static global coordinates \( t, x \)
A light ray can travel to infinity and back in a finite time
This is related to the (conjectured) instability of AdS
– a wave packet can bounce back many times to the center, and in the end always collapses to a black hole
– smaller amplitude $\rightarrow$ more bounces needed
– demonstrated numerically for a spherically symmetric massless scalar field coupled to gravity

AdS is linearly stable
positive mass theorem $\rightarrow$ cannot decay
– adding a small energy, a small black hole may form
– energy cannot disperse at infinity
– it is like a bounding box
– all natural boundary conditions are reflective
Asymptotically AdS spacetimes

Definition using Penrose’s conformal treatment of infinity


Also gives definition for **conserved quantities:**
  - total mass and 3 components of angular momentum
Weakly asymptotically AdS spacetimes ($\Lambda < 0$)

Manifold $\mathcal{M}$, metric $g_{\mu\nu}$, vacuum solution of Einstein’s equations

$(\mathcal{M}, g_{\mu\nu})$ is weakly asymptotically AdS if

(i) there is a manifold $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ with boundary $\mathcal{I}$, and a diffeomorphism $\psi$ from $\mathcal{M}$ onto $\tilde{\mathcal{M}} \setminus \mathcal{I}$, such that $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

(ii) $\Omega = 0$ on $\mathcal{I}$, but its gradient is nonzero

(iii) $\mathcal{I}$ is topologically $S^2 \times \mathbb{R}$

Infinity is part of the new manifold, it is at finite coordinates, and finite distance with respect to the unphysical metric

Einstein’s equations $\rightarrow \mathcal{I}$ timelike
AdS is asymptotically AdS

\[
\Omega = \frac{\cos x}{L}, \quad \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \psi \text{ is identity map}
\]

\[
d\tilde{s}^2 = -dt^2 + dx^2 + \sin^2 x \, d\Omega^2
\]

General metric: \( g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} + h_{\mu\nu} \), we use \( t, x \) coordinates

- weekly asymptotically AdS if \( h_{\mu\nu} \) diverges as \( (\frac{\pi}{2} - x)^{-2} \)
- asymptotically AdS if \( h_{\mu\nu} \) diverges as \( (\frac{\pi}{2} - x)^{-1} \)
Conformal structure of infinity

If a spacetime is asymptotically AdS with some $\Omega$, then $\bar{\Omega} = \omega \Omega$ is equally good, where $\omega$ any function for which $\omega \neq 0$ on $\mathcal{I}$

$\longrightarrow$ only the conformal geometry of $\mathcal{I}$ is meaningful

For exact AdS, with our choice of coordinates, and choice of $\Omega$

$$d\tilde{s}^2 = -dt^2 + dx^2 + \sin^2 x d\Omega^2$$

$\longrightarrow$ the induced metric on $\mathcal{I}$, at $x = \frac{\pi}{2}$, is $d\tilde{s}_0^2 = -dt^2 + d\Omega^2$

it is the flat metric on $S^2 \times \mathbb{R}$

$-\text{with general coordinates and general $\Omega$}$

$d\tilde{s}_0^2$ can be any conformally flat metric

For general weakly asymptotically AdS spacetimes the induced metric on $\mathcal{I}$ can be anything — this is a problem, since

Asymptotic symmetries correspond to

asymptotic Killing vector fields on $\mathcal{M}$

and they correspond to conformal Killing vector fields on $\mathcal{I}$
Definition of asymptotically AdS spacetimes

AdS has the symmetries corresponding to the AdS group $O(3, 2) \rightarrow 10$ conformal Killing vector fields on $\mathcal{I}$

For general weakly asymptotically AdS metric $\mathcal{I}$ may have no conformal Killing vector fields at all

To each conformal Killing field a conserved quantity belongs
– should the number of conserved quantities depend on which metric we choose?

$(\mathcal{M}, g_{\mu\nu})$ is asymptotically AdS if
– it is weakly asymptotically AdS [ points (i), (ii), (iii) ]
– (iv) the metric on $\mathcal{I}$ induced by $\tilde{g}_{\mu\nu}$ is conformally flat

Equivalent to the condition that the symmetry group is the AdS group
In order to have regular quantities at \( \mathcal{I} \) we work with quantities belonging to the unphysical metric \( \tilde{g}_{\mu\nu} \).

Einstein’s equations \( \rightarrow \) Weyl tensor \( \tilde{C}_{\mu\nu\rho\sigma} = 0 \) on \( \mathcal{I} \).

\( \rightarrow \) we can define the leading order asymptotic Weyl at \( \mathcal{I} \) by

\[
K_{\mu\nu\rho\sigma} = \lim_{\mathcal{I} \to \mathcal{I}} \frac{1}{\Omega} \tilde{C}_{\mu\nu\rho\sigma}
\]

The one-form \( n_{\mu} = \nabla_{\mu} \Omega \) is normal to \( \mathcal{I} \), and \( \tilde{g}^{\mu\nu} n_{\mu} n_{\nu} = 1/L^2 \) on \( \mathcal{I} \), where \( L^2 = -3/\Lambda \).

The electric part of the leading order asymptotic Weyl tensor is defined as

\[
E_{\mu\nu} = L^2 K_{\mu\rho\nu\sigma} n^\rho n^\sigma
\]

\( E_{\mu\nu} n^\nu = 0 \rightarrow \) it is a tensor in \( \mathcal{I} \) – it is symmetric and trace-free.
Conserved quantities

If there are no matter fields, then \( \tilde{D}^\mu \mathcal{E}_{\mu\nu} = 0 \), where \( \tilde{D}_\mu \) belongs to the metric \( \tilde{\gamma}_{\mu\nu} \) induced by \( \tilde{g}_{\mu\nu} \) on \( \mathcal{I} \). \( \mathcal{I} \) is \( S^2 \times R \), and has 10 conformal Killing fields \( \xi^\mu \).

For any choice of \( \xi^\mu \) and for any 2-sphere cross section \( C \) of \( \mathcal{I} \), a conserved quantity can be defined by

\[
Q_\xi = -\frac{L}{8\pi} \oint_C \mathcal{E}_{\mu\nu} \xi^\mu u^\nu d\tilde{S}
\]

where \( d\tilde{S} \) is the volume element on \( C \), and \( u^\nu \) is the unit normal to \( C \), both with respect to the metric \( \tilde{\gamma}_{\mu\nu} \).
Mass and angular momentum

The conserved quantity \( \xi = -\frac{L}{8\pi} \oint_C \mathcal{E}_{\mu\nu} \xi^\mu u^\nu d\tilde{S} \)

- is independent of the choice of the hypersurface \( C \)
- is independent of the choice of the conformal factor \( \Omega \)

The simplest choice for \( C \) is to take a constant \( t \) section

Coordinates on \( \mathcal{I} \) are \( x^\mu = (t, \theta, \phi) \)

Choices for the conformal Killing field \( \xi^\mu \):

\[
\xi^\mu = \left( \frac{1}{L}, 0, 0 \right) \quad \rightarrow \quad Q_\xi \text{ gives the total mass}
\]

\[
\xi^\mu = (0, 0, 1) \quad \xi^\mu = (0, -\sin \phi, -\cot \theta \cos \phi) \quad \xi^\mu = (0, \cos \phi, -\cot \theta \sin \phi)
\]

\begin{align*}
\text{3 components of} & \quad \text{angular momentum} \\
\text{they are time independent}
\end{align*}
Localized time-periodic vacuum solutions formed by gravitational waves, with regular center and no horizon for $\Lambda < 0$

- typical size given by the length-scale $L = \sqrt{-\frac{3}{\Lambda}}$

$\Lambda < 0$ makes the formation of geons easier
- there are solutions without high-frequency waves even in the small amplitude linear limit

In this sense, they are similar to the sine-Gordon breathers

Also similar to spherically symmetric scalar AdS breathers formed by a self-gravitating massless real Klein-Gordon field


There are no spherically symmetric geon solutions for vacuum
Consider a one-parameter family of solutions depending on a parameter \( \varepsilon \), and expand the metric as

\[
g_{\mu\nu} = \sum_{k=0}^{\infty} \varepsilon^k g_{\mu\nu}^{(k)}
\]

where \( g_{\mu\nu}^{(0)} \) is the AdS metric

\[
ds^2(0) = \frac{L^2}{\cos^2 x} \left[ -dt^2 + dx^2 + \sin^2 x(\theta^2 + \sin^2 \theta d\phi^2) \right]
\]

\( g_{\mu\nu}^{(0)} \) has components that diverge as \((\frac{\pi}{2} - x)^{-2}\) at infinity

We require that for \( k \geq 1 \) all \( g_{\mu\nu}^{(k)} \) diverge at most as \((\frac{\pi}{2} - x)^{-1}\)

\( \rightarrow \) metric induced at \( \mathcal{I} \) remains the same as for AdS

\( \rightarrow g_{\mu\nu} \) is asymptotically AdS
The physical frequency $\bar{\omega}$ of geon families generally depend on $\varepsilon$ – in the small $\varepsilon$ limit $\bar{\omega}$ approaches integer values.

For technical simplicity we use a time coordinate for which the coordinate frequency $\omega$ remains an $\varepsilon$ independent integer – for this we have to rescale the time coordinate as $t \rightarrow \alpha(\varepsilon)t$.

Since $g^{(0)}_{tt} = -\frac{L^2}{\cos^2 x}$, it follows that the asymptotic behavior of the total metric component $g_{tt}$ becomes $\varepsilon$ dependent.

$$\lim_{x \rightarrow \frac{\pi}{2}} \left[ g_{tt} \left( \frac{\pi}{2} - x \right)^2 \right] = -\nu$$

where $\nu$ depends on $\varepsilon$, but independent of the coordinates.

The relation between the two frequencies becomes $\bar{\omega} = \frac{\omega}{\sqrt{\nu}}$. 
Decomposition along spheres

We decompose $2 + 2$ along the symmetry spheres of the background AdS

$a, b, c \ldots = 1, 2$, $i, j, k \ldots = 3, 4$

- coordinates along the time-radius plane: $x^a = (x^1, x^2) = (t, x)$
- coordinates along the symmetry spheres: $x^i = (x^3, x^4) = (\theta, \phi)$

We decompose AdS as

$$ds^2(0) = g_{ab}dx^a dx^b + r^2 \gamma_{ij} dx^i dx^j$$

where

$$g_{ab} = \frac{L^2}{\cos^2 x} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \quad r = L \tan x , \quad \gamma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$
Spherical harmonic decomposition

We use real spherical harmonics $S_{lm}$
- defined for $l \geq 0$ and $-l \leq m \leq l$ integers
- $\phi$ dependence is $\cos(m\phi)$ for $m \geq 0$, and $\sin(|m|\phi)$ for $m < 0$

Tensors can be decomposed into
- scalar part
- vector part
- tensor part – only for higher than 4 dimensional spacetimes

Vector spherical harmonics has components

$$V_{(lm)\theta} = \frac{1}{\sqrt{l(l+1)}} \frac{1}{\sin \theta} \frac{\partial S_{lm}}{\partial \phi}, \quad V_{(lm)\phi} = \frac{-1}{\sqrt{l(l+1)}} \sin \theta \frac{\partial S_{lm}}{\partial \theta}$$

Perturbations for each $l, m$ can be considered separately
- they are only coupled by lower order terms in the $\varepsilon$ expansion
Vector-type perturbations of the metric

We consider vector-type first, because it is technically simpler

Even if we start with scalar-type perturbations at linear order in $\varepsilon$, vector-type components appear at $\varepsilon^2$ order

$\nabla_{(lm)i}$ components at $\varepsilon^k$ order in the expansion $(a, b = 1, 2; i, j = 3, 4)$

\[
g^{(k)}_{ab} = 0 \, , \quad g^{(k)}_{ai} = Z_a \nabla_i \, , \quad g^{(k)}_{ij} = 0
\]

Only two unknown functions: $Z_a = (Z_t, Z_x)$
  - they depend on the coordinates $x^a = (t, x)$

The $l = 1$ spherical harmonics have to be considered separately
  - for the linear order only the trivial solution $Z_a = 0$
  - for higher order determines the angular momentum
Scalar function determining vector-type perturbations

For $l \geq 2$, from Einstein’s equations follows that there exists a scalar function $\phi$ such that

$$Z_t = \frac{\partial \phi}{\partial x} + \tilde{Z}_t, \quad Z_x = \frac{\partial \phi}{\partial t}$$

where $\tilde{Z}_t$ is an already known function fixed by lower order perturbations $\rightarrow$ zero at linear order

Defining a rescaled scalar function by

$$\phi = r\Phi, \quad r = L \tan x$$

from Einstein’s equations follows that

$$-\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \Phi}{\partial x^2} - \frac{l(l+1)}{\sin^2 x} \Phi = \frac{\bar{\Phi}}{\sin^2 x}$$

where $\bar{\Phi}$ is a known function of $t, x$ determined at lower order in $\varepsilon$
Scalar-type perturbations of the metric

For each $l \geq 2$ and $m$, for scalar-type perturbations also exists a scalar function $\phi = r\Phi$, which satisfies the same equation

\[- \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \Phi}{\partial x^2} - \frac{l(l + 1)}{\sin^2 x} \Phi = \frac{\Phi}{\sin^2 x}\]

The metric perturbations can be obtained from $\Phi$ as follows:

First, calculate the quantities $Z_{ab}$ and $Z$

\[
Z_{tt} = \partial_t^2 \phi - \tan x \partial_x \phi + \frac{\phi}{\cos^2 x} + \tilde{Z}_{tt}
\]

\[
Z_{tx} = \partial_t \partial_x \phi - \tan x \partial_t \phi + \tilde{Z}_{tx}
\]

\[
Z_{xx} = \partial_x^2 \phi - \tan x \partial_x \phi - \frac{\phi}{\cos^2 x} + \tilde{Z}_{xx}
\]

\[
Z = \frac{\cos^2 x}{L^2} (Z_{xx} - Z_{tt}) + \tilde{Z}
\]

where $\tilde{Z}_{ab}$ and $\tilde{Z}$ are determined at lower order in $\varepsilon$
Second step: from $Z_{ab}$ and $Z$ calculate

$$H_L = \frac{r^2}{2} Z, \quad H_{ab} = Z_{ab} - \frac{1}{2} Z g_{ab}$$

The general $S_{lm}$ scalar-type metric perturbations at $\epsilon^k$ order are $(a, b = 1, 2; \ i, j = 3, 4)$

$$g^{(k)}_{ab} = H_{ab} S, \quad g^{(k)}_{ai} = 0, \quad g^{(k)}_{ij} = H_L \gamma_{ij} S$$

The $l = 0, 1$ scalar-type perturbations have to be treated separately
- no generating scalar function
- only gives gauge modes at linear order in $\epsilon$
- $l = 0$ mode determines the mass at higher orders

We have made the most natural gauge choice in the higher order generalization of the gauge invariant formalism of Mukohyama-Kodama-Ishibashi-Seto-Wald
For $l \geq 2$ all scalar or vector perturbations are determined by

$$\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \Phi}{\partial x^2} - l(l + 1) \frac{\Phi}{\sin^2 x} = \frac{\bar{\Phi}}{\sin^2 x}$$

where $\bar{\Phi}$ is given from lower order in $\varepsilon$ results.

The boundary conditions at infinity are different in the two cases:

The generated metric perturbation will be asymptotically AdS if

- for vector-type perturbations $\lim_{x \to \frac{\pi}{2}} \Phi = 0$
- for scalar-type perturbations $\lim_{x \to \frac{\pi}{2}} \frac{d\Phi}{dx} = 0$
Periodic solutions at linear order

There is no inhomogeneous source term: $\bar{\Phi} = 0$

Search solutions in the form $\Phi = p(x) \cos(\omega t - \delta)$

Centrally regular and asymptotically AdS solutions only exist:

- scalar-type perturbations: $\omega = l + 1 + 2n$, $n \geq 0$ integer

$$p(x) = \frac{\alpha}{L} \sin^{l+1} x \frac{n!}{(l + \frac{3}{2})_n} P_n^{(l+\frac{1}{2}, -\frac{1}{2})}(\cos(2x))$$

- vector-type perturbations: $\omega = l + 2 + 2n$, $n \geq 0$ integer

$$p(x) = \frac{\alpha}{L} \sin^{l+1} x \cos x \frac{n!}{(l + \frac{3}{2})_n} P_n^{(l+\frac{1}{2}, \frac{1}{2})}(\cos(2x))$$

where the Pochhammer's Symbol is $(c)_n = \Gamma(c + n)/\Gamma(c)$ and $P_n^{\alpha,\beta}(z)$ are Jacobi polynomials

$n$ gives the number of radial nodes (zero crossings)
Inhomogeneous equation at higher orders in $\varepsilon$

$$-rac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \Phi}{\partial x^2} - \frac{l(l+1)}{\sin^2 x} \Phi = \frac{\bar{\Phi}}{\sin^2 x}$$

homogeneous solutions
with frequency $\omega$
scalar-type: $\omega = l + 1 + 2n$
vector-type: $\omega = l + 2 + 2n$

source terms
of the type
$\bar{\Phi} = p_0(x) \sin(\omega_s t)$ or
$\bar{\Phi} = p_0(x) \cos(\omega_s t)$

If $\omega \neq \omega_s$ for all $n \geq 0$ integers, there are always time-periodic solutions which are asymptotically AdS and have a regular center

If $\omega = \omega_s$ for some $n$, it is the resonant case
- generally, all regular asymptotically AdS solutions are blow-up solutions of the type $t \cos(\omega t)$
- time-periodic solutions only exist if a consistency condition holds

G. Fodor: AdS geons 29/41
Consistency conditions

In the resonant case, time-periodic centrally regular asymptotically AdS solutions only exist, if a **consistency condition** holds on the source term \( \Phi = p_0(x) \sin(\omega_s t) \)

\[
\int_0^{\pi/2} f_{l,n}(x)p_0(x)dx = 0
\]

\( f_{l,n}(x) \) is a function determined by the homogeneous solutions

The consistency condition determines

- the change of physical frequency \( \bar{\omega} \) as a function of \( \varepsilon \)
- ratio of the modes included at linear order

If the consistency conditions cannot be satisfied

\( \Rightarrow \) terms with linearly increasing amplitude \( t \cos(\omega t) \)

\( \rightarrow \) shift of energy to higher frequency modes

\( \dashrightarrow \) turbulent instability \( \sim \) black hole formation
Helically symmetric rotating solution — linear order

The smallest frequency is for the $l = 2, n = 0$ nodeless scalar-type solution: $\omega = l + 1 + 2n = 3$

- then the scalar function is $\Phi = \frac{\alpha}{L} \sin^3 x \cos(\omega t - \delta)$

We take a combination of the $m = 2$ and $m = -2$ modes

$$S_{2,2} = \frac{1}{4} \sqrt{\frac{15}{\pi}} \sin^2 \theta \cos(2\phi), \quad S_{2,-2} = \frac{1}{4} \sqrt{\frac{15}{\pi}} \sin^2 \theta \sin(2\phi)$$

with time shifted phases $\delta = 0$ and $\delta = \frac{\pi}{2}$ and same amplitude

Even at higher orders, all time dependence will be through terms $\cos[k(3t - 2\phi)]$ and $\sin[k(3t - 2\phi)]$

$\longrightarrow$ helical symmetry with Killing vector $\frac{d}{dt} + \frac{3}{2} \frac{d}{d\phi}$

This is the only $\omega = 3$ solution with nonzero angular momentum and reflection symmetry with respect to the equatorial plane
We constructed the \((l, m, n) = (2, \pm 2, 0)\) helically symmetric solution up to fourth order in \(\varepsilon\)

- had to solve consistency conditions at orders \(\varepsilon^3\) and \(\varepsilon^5\)
  
  to fix the unspecified constants

\(\rightarrow\) we have a one-parameter family of solutions

The reparametrization freedom \(\varepsilon \rightarrow f(\varepsilon)\) is fixed by setting the angular momentum \(J = \frac{27\pi L^2}{128}\varepsilon^2\)

This solution was studied first analytically by


and then numerically by

The expansion of the physical frequency is

\[ \bar{\omega} L = 3 \left( 1 + \omega_1 \frac{J}{L^2} + \omega_2 \frac{J^2}{L^4} + \ldots \right) \]

where

\[ \omega_1 = -\frac{4901}{3780\pi} \approx -0.412708 \]
\[ \omega_2 = \frac{7823862709549425\pi^2 - 76880912765261056}{73229764608000\pi^2} \approx 0.466991 \]

The expansion of the total mass is

\[ \frac{M}{L} = \frac{3}{2} \left( \frac{J}{L^2} + \frac{\omega_1}{2} \frac{J^2}{L^4} + \frac{\omega_2}{3} \frac{J^3}{L^6} + \ldots \right) \]

They satisfy the identity \[ \frac{dM}{dJ} = \frac{\bar{\omega}}{2} \] ← first law of geon dynamics
Frequency — angular momentum

\[ J^{AMD}/L^2 \]

- 3rd order
- 5th order
- Numerical

\[ \omega L/m \]
Mass — angular momentum

![Graph showing mass vs. angular momentum](image)
Numerical method

KADATH library – multi-domain spectral method
– developed by Philippe Grandclément at Paris Observatory
– Ads geons – PhD topic of Grégoire Martinon, 2017

Maximal slicing \((K = 0)\) and harmonic coordinates in space
– De-Turck method

Start from a linearized solution – increase the amplitude in steps
– typical resolution: radial 37, angular 9 × 9
– typical running time: several days on hundreds of processors
Geons without radial nodes

\[(l, m, n)\]

\[(2, 2, 0)\]

\[(4, 4, 0)\]

\[(6, 6, 0)\]
Geons with one radial node

Trying to construct a helically symmetric geon from the 
\((l, m, n) = (2, \pm 2, 1)\) linear mode

\(\rightarrow\) contradiction at \(\varepsilon^3\) order

\(\rightarrow\) no one-parameter family of solutions is generated

- the frequency of this mode is \(\omega = l + 1 + n = 5\)

- the angular frequency is \(\omega_a = \frac{\omega}{m} = \frac{5}{2}\)

There are two other linear modes with the same frequencies:

\((l, m, n) = (4, \pm 2, 0)\) scalar mode: \(\omega = l + 1 + n = 5\) , \(\omega_a = \frac{5}{2}\)

\((l, m, n) = (3, \pm 2, 0)\) vector mode: \(\omega = l + 2 + n = 5\) , \(\omega_a = \frac{5}{2}\)

none of them generates a one-parameter family of solutions
Combining three modes at linear order

\((2, \pm 2, 1)\) with amplitude \(\alpha\)
\((4, \pm 2, 0)\) with amplitude \(\beta\)
\((3, \pm 2, 0)\) with amplitude \(\gamma\)

We get three conditions at \(\varepsilon^3\) order

\(\rightarrow\) can be transformed to a 13-th degree polynomial,

it has 3 real and 10 complex roots

\(\rightarrow\) we have 3 one-parameter families of solutions

The ratios of the amplitudes are

\[
\begin{array}{c|c|c}
 & \frac{\beta}{\alpha} & \frac{\gamma}{\alpha} \\
\hline
\text{family I.} & -0.00286074 & 0.154618 \\
\text{family II.} & 0.0367439 & -1.67172 \\
\text{family III.} & 1.07086 & 1.39907 \\
\end{array}
\]

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Angular frequency — angular momentum

\[ \omega L/m \]

\[ J^{AMD}/L^2 \]

I

II

III

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Outlook – things to do

- Investigate non-rotating AdS geons
  - with or without axial symmetry
  - analytically and numerically
- Improve numerical method to reach maximal mass geons
- Study the stability of geons
  - $3 + 1$ dimensional time-evolution code
- Construction of asymptotically flat geons